19. Logic constraints, integer variables

- If-then constraints
- Generalized assignment problems
- Logic constraints
- Modeling a restricted set of values
- Sudoku!
If-then constraints

A single simple trick (with suitable adjustments) can help us model a great variety of if-then constraints

The trick

- We’d like to model the constraint: if $z = 0$ then $a^T x \leq b$.
- Let $M$ be an upper bound for $a^T x - b$.
- Write: $a^T x - b \leq Mz$
- If $z = 0$, then $a^T x - b \leq 0$ as required. Otherwise, we get $a^T x - b \leq M$, which is always true.
If-then constraints

Slight change: if \( z = 1 \) then \( a^T x \leq b \)

- Again, let \( M \) be an upper bound for \( a^T x - b \)
- Write: \( a^T x - b \leq M(1 - z) \)

Reversed inequality: if \( z = 0 \) then \( a^T x \geq b \)

- Write constraint as \(-a^T x + b \leq 0\)
- Let \( m \) be an upper bound on \(-a^T x + b\)
- Write: \(-a^T x + b \leq mz\). Same as: \( a^T x - b \geq -mz\)
- Note: \(-m\) is a lower bound on \( a^T x - b\).
If-then constraints

The converse: if $a^T x \leq b$ then $z = 1$

- Equivalent to: if $z = 0$ then $a^T x > b$ (contrapositive).

- The strict inequality is not really enforceable. Instead, write: if $z = 0$ then $a^T x \geq b + \varepsilon$ where $\varepsilon$ is small.

- Let $m$ be a lower bound for $a^T x - b$ and we obtain the equivalent constraint: $a^T x - b \geq mz + \varepsilon(1 - z)$

- If $z = 0$, we get $a^T x \geq b + \varepsilon$, as required.
  Otherwise, we get: $a^T x - b \geq m$, which is always true.

- **Note:** If $a$, $x$, $b$ are integer-valued, we may set $\varepsilon = 1$. 
If-then constraints (summary)

<table>
<thead>
<tr>
<th>Logic statement</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>if $z = 0$ then $a^T x \leq b$</td>
<td>$a^T x - b \leq Mz$</td>
</tr>
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<td>if $z = 0$ then $a^T x \geq b$</td>
<td>$a^T x - b \geq mz$</td>
</tr>
<tr>
<td>if $z = 1$ then $a^T x \leq b$</td>
<td>$a^T x - b \leq M(1 - z)$</td>
</tr>
<tr>
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<td>$a^T x - b \geq m(1 - z)$</td>
</tr>
<tr>
<td>if $a^T x \leq b$ then $z = 1$</td>
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</tr>
<tr>
<td>if $a^T x \geq b$ then $z = 1$</td>
<td>$a^T x - b \leq Mz - \varepsilon(1 - z)$</td>
</tr>
<tr>
<td>if $a^T x \leq b$ then $z = 0$</td>
<td>$a^T x - b \geq m(1 - z) + \varepsilon z$</td>
</tr>
<tr>
<td>if $a^T x \geq b$ then $z = 0$</td>
<td>$a^T x - b \leq M(1 - z) - \varepsilon z$</td>
</tr>
</tbody>
</table>

Where $M$ and $m$ are upper and lower bounds on $a^T x - b$.  

19-5
Return to fixed costs and lower bounds

- Modeling a fixed cost: if \( x > 0 \) then \( z = 1 \).
  - Use the contrapositive: if \( z = 0 \) then \( x \leq 0 \).
  - Apply the 1\(^{\text{st}}\) rule on Slide 19-5.

- Modeling a lower bound: either \( x = 0 \) or \( x \geq m \).
  - Equivalent to: if \( x > 0 \) then \( x \geq m \).
  - Equivalent to the following two logical constraints: if \( x > 0 \) then \( z = 1 \), and if \( z = 1 \) then \( x \geq m \).
  - The first one is a fixed cost (see above)
  - The second one is the 4\(^{\text{th}}\) rule on Slide 19-5.
Generalized assignment problems (GAP)

- Set of machines: $\mathcal{M} = \{1, 2, \ldots, m\}$ that can perform jobs. (think of these as the facilities in the facility problem)
- Machine $i$ has a fixed cost of $h_i$ if we use it at all.
- Machine $i$ has a capacity of $b_i$ units of work (this is new!)

- Set of jobs: $\mathcal{N} = \{1, 2, \ldots, n\}$ that must be performed. (think of these as the customers in the facility problem)
- Job $j$ requires $a_{ij}$ units of work to be completed if it is completed on machine $i$.
- Job $j$ will cost $c_{ij}$ if it is completed on machine $i$.
- Each job must be assigned to exactly one machine.
GAP model

minimize \[ \sum_{i \in M} h_i z_i + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} \] (fixed cost + assignment cost)

subject to:
\[ \sum_{i \in M} x_{ij} = 1 \quad \forall j \in N \] (one machine per job)
\[ \sum_{j \in N} a_{ij} x_{ij} \leq b_i \quad \forall i \in M \] (work budget)
\[ x_{ij} \leq z_i \quad \forall i \in M, j \in N \] (if \( x_{ij} > 0 \) then \( z_i = 1 \))
\[ x_{ij}, z_i \in \{0, 1\} \quad \forall i \in M, j \in N \] (all binary!)

- \( z_i = 1 \) if machine \( i \) is used, and
- \( x_{ij} = 1 \) if job \( j \) is performed by machine \( i \).
- **Note:** many choices possible for \( M_i \) and aggregations.
New constraints

Let’s make GAP more interesting...

1. If you use $k$ or more machines, you must pay a penalty of $\lambda$.

2. If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.

3. If you operate both machines 1 and 2, then machine 3 must be operated at 40% of its capacity.

4. Each job $j \in \mathcal{N}$ has a duration $d_j$. Minimize the time we have to wait before all jobs are completed. (This is called the makespan).
If you use $k$ or more machines, you must pay a penalty of $\lambda$.

- Using $k$ or more machines is equivalent to saying that
  
  $$z_1 + z_2 + \cdots + z_m \geq k$$
  
- Let $\delta_1 = 1$ if we incur the penalty. We now have the if-then constraint: if $\sum_{i \in M} z_i \geq k$ then $\delta_1 = 1$.

- Use the 6th rule on Slide 19-5 and obtain:
  
  $$\sum_{i \in M} z_i \leq m\delta_1 + (k - 1)(1 - \delta_1)$$
  
- add $\lambda\delta_1$ to the cost function.
If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.

- Operating machine 1 or machine 2: \( z_1 + z_2 \geq 1 \).
- Not operating machines 3 and 4: \( z_3 + z_4 \leq 1 \).
- We must model \( z_1 + z_2 \geq 1 \implies z_3 + z_4 \leq 1 \)
  - Same trick as before: model this in two steps:
    \( z_1 + z_2 \geq 1 \implies \delta_2 = 1 \text{ and } \delta_2 = 1 \implies z_3 + z_4 \leq 1 \)
  - First follows from 6\(^{th}\) rule on Slide 19-5
  - Second follows from 3\(^{rd}\) rule on Slide 19-5
- Result: \( z_1 + z_2 \leq 2\delta_2 \text{ and } z_3 + z_4 + \delta_2 \leq 2 \).
GAP 2 (cont’d)

If you operate either machine 1 or machine 2, you may not operate both machines 3 and 4 at the same time.

We didn’t do anything to ensure that when $z_i = 1$, the machines are actually operating! (we didn’t explicitly disallow paying the fixed cost without using the machine).

- To force the converse as well, include the constraint:
  
  if $z_i = 1$ then $\sum_{j \in \mathcal{N}} x_{ij} \geq 1$

- Use the 4th rule on Slide 19-5.

- Result: $\sum_{j \in \mathcal{N}} x_{ij} \geq z_i$ (for $i = 1, 2, 3, 4$)
If you operate both machines 1 and 2, then machine 3 must be operated at 40% of its capacity.

- Operate both machines 1 and 2: \( z_1 + z_2 \geq 2 \)
- Capacity of machine 3 drops: \( b_3 \) becomes 0.4\( b_3 \).
- Two parts to the implementation:
  - \( z_1 + z_2 \geq 2 \iff \delta_3 = 1 \). (6th rule on Slide 19-5)
  - \( \delta_3 = 1 \iff \sum_{j \in N} a_{3j}x_{3j} \leq 0.4b_3 \). (3rd rule on Slide 19-5)
- Equivalently, just replace \( b_3 \) by: \( b_3(1 - \delta_3) + 0.4b_3\delta_3 \).
Each job $j \in \mathcal{N}$ has a duration $d_j$. Minimize the time we have to wait before all jobs are completed. (the makespan)

- Machine $i$ completes all its jobs in time: $\sum_{j \in \mathcal{N}} x_{ij} d_j$

- Minimax problem (no integer variables needed!)

- Let $t$ be the makespan; $t = \max_{i \in \mathcal{M}} \left( \sum_{j \in \mathcal{N}} x_{ij} d_j \right)$

- Model: minimize $t$ subject to:

$$t \geq \sum_{j \in \mathcal{N}} x_{ij} d_j \quad \text{for all } i \in \mathcal{M}$$
Logic constraints

• A **proposition** is a statement that evaluates to true or false. One example we’ve seen: a linear constraint $a^T x \leq b$.

• We’ll use binary variables $\delta_i$ to represent propositions $P_i$:
  
  $\delta_i = \begin{cases} 
  1 & \text{if proposition } P_i \text{ is true} \\
  0 & \text{if proposition } P_i \text{ is false} 
  \end{cases}$

  The term for this is that $\delta_i$ is an **indicator variable**.

How can we turn logical statements about the $P_i$’s into algebraic statements involving the $\delta_i$’s?

Some standard notation:

- $\lor$ means “or”
- $\land$ means “and”
- $\neg$ means “not”
- $\implies$ means “implies”
- $\iff$ means “if and only if”
- $\oplus$ means “exclusive or”
Boolean algebra

Basic definitions:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ∧ Q</th>
<th>P ∨ Q</th>
<th>P ⊕ Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
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</tbody>
</table>

Useful relationships:

- \( \neg(P_1 \land \cdots \land P_k) = \neg P_1 \lor \cdots \lor \neg P_k \)
- \( \neg(P_1 \lor \cdots \lor P_k) = \neg P_1 \land \cdots \land \neg P_k \)
- \( P \land (Q \lor R) = (P \land Q) \lor (P \land R) \)
- \( P \lor (Q \land R) = (P \lor Q) \land (P \lor R) \)
- \( P \oplus Q = (P \land \neg Q) \lor (\neg P \land Q) \)
## Logic to algebra

<table>
<thead>
<tr>
<th>Statement</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg P_1$</td>
<td>$\delta_1 = 0$</td>
</tr>
<tr>
<td>$P_1 \lor P_2$</td>
<td>$\delta_1 + \delta_2 \geq 1$</td>
</tr>
<tr>
<td>$P_1 \oplus P_2$</td>
<td>$\delta_1 + \delta_2 = 1$</td>
</tr>
<tr>
<td>$P_1 \land P_2$</td>
<td>$\delta_1 = 1, \delta_2 = 1$</td>
</tr>
<tr>
<td>$\neg (P_1 \lor P_2)$</td>
<td>$\delta_1 = 0, \delta_2 = 0$</td>
</tr>
<tr>
<td>$P_1 \implies P_2$</td>
<td>$\delta_1 \leq \delta_2$ (equivalent to: $(\neg P_1) \lor P_2$)</td>
</tr>
<tr>
<td>$P_1 \implies (\neg P_2)$</td>
<td>$\delta_1 + \delta_2 \leq 1$ (equivalent to: $\neg (P_1 \land P_2)$)</td>
</tr>
<tr>
<td>$P_1 \iff P_2$</td>
<td>$\delta_1 = \delta_2$</td>
</tr>
<tr>
<td>$P_1 \implies (P_2 \land P_3)$</td>
<td>$\delta_1 \leq \delta_2, \delta_1 \leq \delta_3$</td>
</tr>
<tr>
<td>$P_1 \implies (P_2 \lor P_3)$</td>
<td>$\delta_1 \leq \delta_2 + \delta_3$</td>
</tr>
<tr>
<td>$(P_1 \land P_2) \implies P_3$</td>
<td>$\delta_1 + \delta_2 \leq 1 + \delta_3$</td>
</tr>
<tr>
<td>$(P_1 \lor P_2) \implies P_3$</td>
<td>$\delta_1 \leq \delta_3, \delta_2 \leq \delta_3$</td>
</tr>
<tr>
<td>$P_1 \land (P_2 \lor P_3)$</td>
<td>$\delta_1 = 1, \delta_2 + \delta_3 \geq 1$</td>
</tr>
<tr>
<td>$P_1 \lor (P_2 \land P_3)$</td>
<td>$\delta_1 + \delta_2 \geq 1, \delta_1 + \delta_3 \geq 1$</td>
</tr>
</tbody>
</table>
## More logic to algebra

<table>
<thead>
<tr>
<th>Statement</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 \lor P_2 \lor \cdots \lor P_k$</td>
<td>$\sum_{i=1}^{k} \delta_i \geq 1$</td>
</tr>
<tr>
<td>$\left( P_1 \land \cdots \land P_k \right) \implies \left( P_{k+1} \lor \cdots \lor P_n \right)$</td>
<td>$\sum_{i=1}^{k} (1 - \delta_i) + \sum_{i=k+1}^{n} \delta_i \geq 1$</td>
</tr>
<tr>
<td>at least $k$ out of $n$ are true</td>
<td>$\sum_{i=1}^{n} \delta_i \geq k$</td>
</tr>
<tr>
<td>exactly $k$ out of $n$ are true</td>
<td>$\sum_{i=1}^{n} \delta_i = k$</td>
</tr>
<tr>
<td>at most $k$ out of $n$ are true</td>
<td>$\sum_{i=1}^{n} \delta_i \leq k$</td>
</tr>
<tr>
<td>$P_n \iff (P_1 \lor \cdots \lor P_k)$</td>
<td>$\sum_{i=1}^{k} \delta_i \geq \delta_n, \delta_n \geq \delta_j, j = 1, \ldots, k$</td>
</tr>
<tr>
<td>$P_n \iff (P_1 \land \cdots \land P_k)$</td>
<td>$\delta_n + k \geq 1 + \sum_{i=1}^{k} \delta_i, \delta_j \geq \delta_n, j = 1, \ldots, k$</td>
</tr>
</tbody>
</table>
Modeling a restricted set of values

• We may want variable $x$ to only take on values in the set \{\(a_1, \ldots, a_m\)\}.

• We introduce binary variables $y_1, \ldots, y_m$ and the constraints

\[
x = \sum_{j=1}^{m} a_j y_j, \quad \sum_{j=1}^{m} y_j = 1, \quad y_j \in \{0, 1\}
\]

• $y_i$ serves to select which $a_i$ will be selected.

• The set of variables $\{y_1, y_2, \ldots, y_m\}$ is called a special ordered set (SOS) of variables.
Example: building a warehouse

• Suppose we are modeling a facility location problem in which we must decide on the size of a warehouse to build.

• The choices of sizes and associated cost are shown below:

<table>
<thead>
<tr>
<th>Size</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>20</td>
<td>180</td>
</tr>
<tr>
<td>40</td>
<td>320</td>
</tr>
<tr>
<td>60</td>
<td>450</td>
</tr>
<tr>
<td>80</td>
<td>600</td>
</tr>
</tbody>
</table>

Warehouse sizes and costs
Example: building a warehouse

- Using binary decision variables $x_1, x_2, \ldots, x_5$, we can model the cost of building the warehouse as
  \[ \text{cost} = 100x_1 + 180x_2 + 320x_3 + 450x_4 + 600x_5. \]

- The warehouse will have size
  \[ \text{size} = 10x_1 + 20x_2 + 40x_3 + 60x_4 + 80x_5, \]

- and we have the SOS constraint
  \[ x_1 + x_2 + x_3 + x_4 + x_5 = 1. \]
What about integers?

- What if \( x \) is an integer, i.e. \( x \in \{1, 2, \ldots, 10\} \)
- First option: use 10 separate variables:
  \[
  x = \sum_{k=1}^{10} k y_k, \quad \sum_{k=1}^{10} y_k = 1, \quad y_k \in \{0, 1\}
  \]
- Another option: use 4 binary variables (less symmetry):
  \[
  x = y_1 + 2y_2 + 4y_3 + 8y_4, \quad 1 \leq x \leq 10, \quad y_k \in \{0, 1\}
  \]

Performance is solver-dependent. If the solver allows integer constraints directly, that’s often the right choice.
Example: Sudoku

- fill grid with numbers \( \{1, 2, \ldots, 9\} \)
- each row and each column contains distinct numbers
- each \(3 \times 3\) cluster contains distinct numbers
Example: Sudoku

- **Decision variables:** $X \in \{0, 1\}^{9 \times 9 \times 9}$ (729 binary variables)
  \[ X_{ijk} = \begin{cases} 1 & \text{if } (i,j) \text{ entry is a } k \\ 0 & \text{otherwise} \end{cases} \]

  Can fill in known entries right away.

- **Basic constraints:** (324 in total)
  - $\sum_{k=1}^{9} X_{ijk} = 1 \forall i,j$ (SOS constraint)
  - $\sum_{i=1}^{9} X_{ijk} = 1 \forall j,k$ (column $j$ contains exactly one $k$)
  - $\sum_{j=1}^{9} X_{ijk} = 1 \forall i,k$ (row $i$ contains exactly one $k$)
  - $\sum_{(i,j) \in C} X_{ijk} = 1 \forall C,k$ (cluster $C$ contains exactly one $k$)

- Much trickier to model using other integer representations!

- **Julia code:** Sudoku.ipynb